Quantum-Hard-Sphere System Equation-of-State Revisited

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Abstract

Analytical equations-of-state for boson and fermion hard-sphere fluids ranging from very low to very high densities are constructed. The fluid branch extrapolations from the exact low-density series expansions for the energy are carried out by incorporating various physical arguments, such as close packing densities and residues. Modified London equations-of-state for the high-density crystalline branch agree very well with the computer simulations and at close packing with certain experimental results at high pressure.

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1. Introduction

The hard-sphere system [1] serves as a first approximation to a many-body system interacting via any pair potential containing a short-ranged repulsive part. The approximation is better at low densities when the particles experience the attractive potential only weakly, as well as at very high densities where the repulsive part predominates. At intermediate densities the attractive potential plays a crucial, essential role. The hard-sphere system also serves as a reference (or zero-order) system in perturbative theories. For instance, in classical statistical physics this scheme is the familiar Thermodynamic Perturbation Theory [2], which very successfully describes classical fluids. Its quantum counterpart, the Quantum Thermodynamic Perturbation Theory (QTPT), has been developed [3] whereby an accurate quantum hard-sphere equation of state for physical (i.e., intermediate) densities plays a critical role in describing quantum fluids such as ^{3}He, ^{4}He, spin-polarized atomic hydrogen H$\downarrow$, nuclear matter, etc. The goal here is to elaborate further upon the efforts reported in [1] to obtain an accurate equation of state for quantum hard sphere fluids at low, intermediate, and high densities.

At very low density the energy E for an N-identical-boson system is known [4] to be given exactly by

$$E/N = \frac{2\pi\hbar^2\rho a}{m}\Big\{1 + C_1(\rho a^3)^{1/2} + C_2\rho a^3\ln(\rho a^3) + \cdots\Big\}, \tag{1}$$

with ρ being the particle number density, m the mass of each boson and where a is the S-wave scattering length [5] of the pair potential between particles, $C_1 = \frac{128}{15\sqrt{\pi}}$ and $C_2 = 8(\frac{4\pi}{3} - \sqrt{3})$. For a hard-sphere system a reduces to the hard-sphere diameter c. This series is clearly not a power series expansion, and is at best an asymptotic series.

For an N-identical-fermion system the corresponding series is [6]

$$
\begin{aligned}
E/N &= \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}\Big\{1 + C_1(k_F a) + C_2(k_F a)^2 \\
&+ [C_3 r_0/2a + C_4 A_1(0)/a^3 + C_5](k_F a)^3 + C_6(k_F a)^4 \ln(k_F a) \\
&+ [C_7 r_0/2a + C_8 A_0''(0)/a^3 + C_9](k_F a)^4 + o[(k_F a)^4]\Big\}, \qquad (2)
\end{aligned}
$$

where the C_j $(j = 1, 2, ..., 9)$, given in [7] for $\nu = 2$ and $\nu = 4$, are dimensionless coefficients depending on ν, the number of intrinsic degrees of freedom of each fermion. (Note: for $\nu = 4$, $C_2 = 0.556610$ instead the incorrect value of 0.566610 quoted in [7], and C_9 is not available for $\nu = 4$) . The Fermi-momentum $\hbar k_F$ is defined through the fermion number density

$$
\rho = N/\Omega = \frac{\nu k_F^3}{6\pi^2}, \qquad (3)
$$

where Ω is the system volume. The quantities a, r_0, and $A_1(0)$ are parameters containing information related to *two*-body scattering due to a central potential $V(r)$. However, $A_0''(0)$ cannot be related to scattering phase shift alone but is potential-shape dependent and can thus be interpreted as the first correction to the static limit, while C_9 has a three-body cluster contribution. Both low-density expansions (1) and (2) break down at moderate and high densities, including the saturation (or equilibrium, zero-pressure) liquid densities of ^{4}He and ^{3}He, or nuclear matter. In Sec. 2 we discuss analytical interpolation between low- and high-densities of the equation of state for bosons and fermions. These interpolations are based on an original equation by London for bosons. In Sec. 3, the series (1) and (2) for hard-sphere systems can be extrapolated to physical and even to close packing densities through the use of Padé and "tailing" [8] extrapolants.

At intermediate densities one has the presumably exact Green Function Monte Carlo (GFMC) computer simulation for the many boson hard-sphere [9] fluid at

four densities. These simulations are a good guide for our analytical extrapolations since they used identical interactions. For fermions, however, the need to satisfy the Pauli exclusion principle has delayed good GFMC calculations for ground state energies. The first results using this method have been reported for the energy of liquid helium-three (^{3}He) [10] where particles interact via the pairwise Aziz [11] potential.

At very high density we expect that the hard sphere system will go to close packing (CP) *independently of statistics*. This packing may be either random or regular. The uncertainty principle implies a second-order pole in the ground-state energy which can be written as

$$E/N \underset{\rho \to \rho_{\rm CP}}{\longrightarrow} A \frac{\hbar^2}{2m} (\rho^{-1/3} - \rho_{\rm CP}^{-1/3})^{-2}, \tag{4}$$

where A, called the residue, is a dimensionless constant, and $\rho_{\rm CP}$ is the appropriate CP density. Using the polyhedron cell method suggested by Yang [12], the value of A has been predicted theoretically [13] to lie within the *rigorous* range

$$1.63 \leq A \leq 27.0 \tag{5}$$

for regular CP (face-centered-cubic, fcc, or hexagonal-close-packing, hcp) by generalizing the exact calculation for a simple cubic (sc) lattice based on three mutually perpendicular linear lattices which gives $A = \pi^2$. On the other hand, the experimental value of A extracted by Cole [14] from high-pressure data of ^{3}He, ^{4}He, H_2 and D_2 systems is $A \simeq 15.7 \pm 0.6$ for the crystalline branch of the equation of state. In Sec. 4 we exhibit the behavior of our equations of state as they go to CP and we calculate the residues. Sec. 5 gives our conclusions.

2. Analytic Interpolations.

The first attempt to represent the ground-state energy per particle of an assembly of $N(\gg 1)$ boson hard spheres for all densities appears to be that of London [15], who proposed the analytical equation of state

$$E/N = \frac{2\pi\hbar^2 c}{m}\frac{1}{(\rho^{-1/3}-\rho_0^{-1/3})^2}\frac{1}{(\rho^{-1/3}+b\,\rho_0^{-1/3})}, \tag{6}$$

where c is the hard-sphere diameter and $b = (2^{5/2}/\pi) - 1$. Here $\rho_0 = \rho_{\rm CP}$ and $\rho_0 \equiv \sqrt{2}/c^3$ is the ultimate density [16] for a system of classical hard-spheres which close packs in a primitive hexagonal, e.g., a face centered cubic (fcc) arrangement. The basic rationale behind (6) is that it reduces smoothly, at both low and high densities, to the well-known limiting expressions

$$E/N \underset{\rho\to 0}{\longrightarrow} (2\pi\hbar^2/m)\rho\, c, \tag{7}$$

$$E/N \underset{\rho\to\rho_0}{\longrightarrow} (\pi^2/2^{1/3})(\hbar^2/2m)(\rho^{-1/3}-\rho_0^{-1/3})^{-2}. \tag{8}$$

The first asymptotic form (7) is the celebrated Lenz [17] term, calculated as the leading correction to the energy due to an "excluded volume" effect. The limiting expression (8) is precisely the kinetic energy of a point particle of mass m inside a spherical cavity of radius $r - c$, where $r = (\sqrt{2}/\rho)^{1/3}$ is the separation between two neighboring spheres. This follows by assuming primitive hexagonal close packing, e.g., hcp or fcc, of the N cavities. Recently, a generalization of the boson London equation (6) was proposed [18] for fermion systems which reads

$$E/N = \mathcal{C}_\nu\,\rho^{2/3} + \left(\frac{\nu-1}{\nu}\right)\frac{2\pi\hbar^2 c}{m}\frac{1}{(\rho^{-1/3}-\rho_0^{-1/3})^2}\frac{1}{[\rho^{-1/3}+b(\nu)\,\rho_0^{-1/3}]}, \tag{9}$$

with

$$\mathcal{C}_\nu = \frac{3\hbar^2}{10m}\left(\frac{6\pi^2}{\nu}\right)^{2/3}, \tag{10}$$

and

$$b(\nu) = [(\nu - 1)/\nu](b + 1) - 1 \quad \underset{\nu \to \infty}{\rightarrow} \quad b, \tag{11}$$

where the limit $\nu = N \to \infty$ corresponds to bosons. The low-density limit of (9) becomes

$$E/N \quad \underset{\rho \to 0}{\rightarrow} \quad \mathcal{C}_\nu \, \rho^{2/3} + \left(\frac{\nu - 1}{\nu}\right) \frac{2\pi\hbar^2}{m} \rho c. \tag{12}$$

The second term on the rhs is the Lenz term for ν-component fermions, and reduces to the boson Lenz term (7) for $\nu \to \infty$. On the other hand, for $\rho \to \rho_0$ one can see that (9) reduces to (8) as it should. From this it follows that the residue for bosons or fermions is the same, and equals $\pi^2/2^{1/3} \simeq 7.83$. Although (9) shows good qualitative behavior for all densities, comparison at intermediate densities with computer simulation GFMC, Variational Monte Carlo (VMC) data, exact Ladder [6] or Variational Fermi Hypernetted-Chain (VFHNC) [19] data, suggests other alternatives to be discussed below.

Also recently it was noted [20] that the derivation of the high-density extreme of the original [15] (boson) London equation (6), and consequently of the generalized [18] (fermion) London equation (9), contains *one fundamental error* related to neglect of the effective two-body mass. Correcting for this gives $b \equiv 2^{3/2}/\pi - 1$ (instead of the $2^{5/2}/\pi - 1$ cited by London) in (6). The new result was designated [20] the *modified London equation*, and continues to satisfy (7) as this is independent of the constant b in (6). The residue A in (4) is now $2^{2/3}\pi^2 \simeq 15.7$ in *full agreement* with Cole's empirical residue [14] quoted below (5). Moreover, this modified London (ML) equation agrees dramatically better than the original London (L) equation with GFMC computer simulation [9] of both fluid and crystalline branches of the boson hard-sphere system. We show this agreement in Fig. 1 where the dimensionless quantity $\varepsilon_0^{-1/2} \equiv \sqrt{2\pi\hbar^2 \rho \, c \, N/m \, E} = [1 - (\rho/\rho_0)^{1/3}]$

$\times \sqrt{1 + b\,(\rho/\rho_0)^{1/3}}$ is plotted as a function of ρ/ρ_0 for the generalized London equation (9), without (dashed curves) and with (full curves) the correct two-body effective mass as discussed in Ref. [20]. The modified boson London curve (ML_∞) agrees very well with the GFMC data [9] for boson hard-spheres; open circles are data for the fluid branch and full circles are data for the solid branch.

3. Low-density series expansions.

We begin with the low density expansions (1) and (2) for the energy and then extrapolate them to intermediate and very high densities through Padé [21] and related approximants.

For boson hard spheres Eq. (1) can be rewritten as

$$E/N = \frac{2\pi\hbar^2}{m}\rho\, c\, e_0(x), \quad \rho = N/\Omega, \quad x \equiv (\rho c^3)^{1/2}, \tag{13}$$

with

$$e_0(x) = 1 + C_1 x + C_2 x^2 \ln x^2 + C_3 x^2 + O(x^3 \ln x^2). \tag{14}$$

Alternatively, one may expand

$$e_0^{-1/2}(x) \simeq 1 + F_1\, x + F_2\, x^2 \ln x^2 + F_3\, x^2 + O(x^3 \ln x^2) \tag{15}$$

for $x \ll 1$, where the F's are expressible in terms of the C's, but C_3 and F_3 and higher-order coefficients are unknown. Values of the F's are given in Table 1. Instead of series-analyzing $e_0(x)$, we prefer to work with $e_0^{-1/2}(x)$ so as to ensure that zeros in the extrapolants to $e_0^{-1/2}(x)$ are second order poles in the energy (13) as expected at CP from (4). Besides the trivial extrapolant, four "tailing" [8] extrapolants to the series (15) with two terms (say, F_1 and F_2) exist and are given in Table I of [22]. However, only the form denoted there by **"i"** is well behaved and acceptable. Here, as well as in [22], well behaved means that the corresponding $\epsilon_0^{-1/2}(x)$ approximant: a) has a zero at some x_{CP} into the interval $0 < x < 1.1$ and b) it doesn't increase

faster than linearly in the interval $0 < x < x_{\rm CP}$, since the energy itself must increase monotonically in this range. The next step is to analyze the *twelve* extrapolants from the series (15) with three (including F_3) terms. The unknown coefficient F_3 was adjusted to ensure agreement with GFMC data. Forms denoted in [22] by VII and XI satisfied the above a) and b) conditions, and although both approximants go across the four GFMC fluid data points the mean square deviation was least with form XI, which we therefore adopt as our best extrapolant. Similar analysis was realized in [22] but they used one half of the correct values for the GFMC energy data which we used to do the fitting. So the ground state energy for boson hard spheres will be represented (symbol $\doteq$) by

$$E/N \doteq \frac{2\pi\hbar^2}{m}\rho\, c\, \varepsilon_0(x), \tag{16}$$

with

$$\varepsilon_0^{-1/2}(x) \doteq \frac{1 + F_2\, x^2 \ln x^2}{1 - F_1\, x - (F_3 - F_1^2)\, x^2} \equiv \mathrm{XI}(x) \tag{17}$$

where $F_3 = -27.956$. In Fig. 2, we graph both the extrapolations $\varepsilon_0^{-1/2}$ as a function of $(\rho/\rho_0)^{1/3}$ for the fluid branches of boson and fermion hard-spheres, as well as the various computer simulations listed there. For bosons, we plot (17) as the upper full curve labeled XI, which fits the fluid GFMC data [9] (open circles). This (fluid branch) curve predicts a random close packing (RCP) density of $\rho = 0.776\,\rho_0$ which is only ten percent below the classical [23] RCP value of $\rho = 0.86\,\rho_0$, which in turn is expected to be the highest CP density also for quantum hard sphere fluids in the metastable region. Since particles at CP are perfectly *localized* they loose their indistinguishability so that all results should be independent of statistics in the limit. A residue of $A = 17.05$ follows from (16) and (17), and thus obeys the bounds (5). Note that this is slightly larger than Cole's experimental residue of 15.7 ± 0.6 as we would expect from arguments to be given below in Sec. 4.

For fermion hard sphere systems (2) can be rewritten as

$$E/N \;=\; \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}\, e_0(x); \qquad x \equiv k_F c, \qquad \rho = N/\Omega = \nu k_F^3/6\pi^2, \tag{18}$$

with

$$\begin{aligned} e_0(x) \simeq 1 \;&+\; C_1\, x \;+\; C_2\, x^2 \;+\; (C_3/3 + C_4/3 + C_5)\, x^3 \\ &+\; C_6\, x^4 \ln x \;+\; (C_7/3 - C_8/3 + C_9)\, x^4 + o(x^4). \end{aligned} \tag{19}$$

For $\nu = 2$, $C_6 = 0$ [7] and (19) simplifies to

$$e_0(x) \;\simeq\; 1 + D_1\, x + D_2\, x^2 + D_3\, x^3 + D_4\, x^4 + o(x^4) \tag{20}$$

for $x \ll 1$, with the $D's$ expressible in terms of the $C's$. Similarly as in the boson case, instead of $e_0(x)$ we shall series-analyze

$$e_0^{-1/2}(x) \simeq 1 + F_1\, x + F_2\, x^2 + F_3\, x^3 + F_4\, x^4 + F_5\, x^5, \tag{21}$$

where the $F_i's$ depend algebraically on the $D_i's$ in a simple manner, but F_5 is unknown. Values of D_i and F_i are given in Table 2. This series is an ordinary power series so that one may apply ordinary Padé approximants. The approximants to (21) with four terms (F_1 to F_4) were discussed in Ref. [24], where it was concluded that the best extrapolant was the $[4/0](x)$ Padé approximant. However, this function does not possess a zero in the region of physical interest, i.e., $0 \le x \le 3.47$, and so the energy does not diverge at any CP. It was thus necessary to introduce the fifth (F_5) term in (21). Its five Padé approximants were analyzed and F_5 adjusted to ensure a zero. The placement of the zero and the approximant were chosen [25] in such a way that the QTPT applied to ^{3}He with the Aziz interatomic potential [11] reproduce the corresponding GFMC [10] data for liquid ^{3}He. Ultimately the best (two-point Padé) extrapolant is

$$\varepsilon_0^{-1/2}(x) \doteq [3//2](x), \tag{22}$$

so that the energy becomes

$$E/N = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}\{[3//2](x)\}^{-2}, \tag{23}$$

with a CP density at $\rho/\rho_0 = 0.732$ and a residue of $A = 4.97$. The resulting coefficient F_5 is listed in Table 2 and (22) is graphed in Fig. 2, as a function of $(\rho/\rho_0)^{1/3}$. As an additional comparison, the results of the L-expansion [26] [27] have also been plotted in Fig. 2. For $\nu = 2$ they agree very well with the [3//2] results.

For $\nu = 4$ fermions, Eq. (19) becomes

$$e_0(x) \simeq 1 + D_1\, x + D_2\, x^2 + D_3\, x^3 + D_4\, x^4 \ln x + D_5\, x^4 + o(x^5) \tag{24}$$

for $x \ll 1$. As with bosons and $\nu = 2$ fermions, we shall analyze instead the series

$$e_0^{-1/2}(x) \simeq 1 + F_1\, x + F_2\, x^2 + F_3\, x^3 + F_4\, x^4 \ln x + F_5\, x^4 \tag{25}$$

with all F_i except F_5 known. Values of D_i and F_i, are given in Table 3. Unlike the case $\nu = 2$, this series is *not* a power series but a mixture of power and logarithmic terms. Its tailing approximants are given in Table III of [22]. We eliminate approximants to $e_0^{-1/2}(x)$ having a pole in the range of physical densities, *i.e.*, from zero to the "empirical" nuclear matter saturation density of $\rho_s = 0.17$ fm^{-3}. This reduces the number of acceptable approximants to five, and from these we select form II and form XII because only these extrapolants have residues within the bounds (5). The energy is finally given by

$$E/N = \frac{3\hbar^2 k_F^2}{10m}\,\varepsilon_0(x), \tag{26}$$

with either

$$\varepsilon_0^{-1/2}(x) \doteq 1 + F_1 x + \frac{F_2 x^2}{1 - F_3/F_2 x - F_4/F_2 x^2 \ln x} \equiv \mathrm{II}(x) \tag{27}$$

or

$$\varepsilon_0^{-1/2}(x) \doteq \frac{1 + (F_1 - F_3/F_2)x + (F_2 - F_1 F_3/F_2)x^2}{1 - (F_3/F_2)x - F_4 x^4 \ln x} \equiv \mathrm{XII}(x) \tag{28}$$

Although both approximants are very similar the residue of extrapolant form II(x) is less than the estimated [13] exact lower bound, which is why we prefer extrapolant form XII(x) as the better extrapolant, and it is graphed in Fig. 2 (dot-dashed curve), as a function of $(\rho/\rho_0)^{1/3}$. This choice of form XII for $\nu = 4$ is reinforced by the agreement with the results of the L-expansion [26] [27]. In spite of the predicted quantum RCP densities denoted by $(\rho/\rho_0)_{\mathrm{CP}}$ in Table 4 lying well below the expected classical RCP density of $\rho/\rho_0 = 0.86$, we hope that the approximants are good at least from zero through nuclear matter saturation density of about $\rho/\rho_0 = 0.008$.

4. Random close packing.

We now analyze the behavior of the energy per particle E/N of hard-sphere systems as they approach CP. By uncertainty-principle arguments, as in (4), we now use ρ_{CP} instead of ρ_0 as the endpoint density so that

$$E_0/N \underset{\rho \to \rho_{\mathrm{CP}}}{\to} A \frac{\hbar^2}{2m} (\rho^{-1/3} - \rho_{\mathrm{CP}}^{-1/3})^{-2} \tag{29}$$

with the residue A depending on the particular geometrical configuration of hard-sphere packing. For regular fcc or hcp close packing $\rho_0 = \sqrt{2}/c^3$, c being the hard sphere diameter. For quantum random close packing (RCP), we expect the packing density to be that of classical RCP, i.e., 0.86 ρ_0.

Although the bounds on A have been found for regular (as opposed to random) close packing such as found in the limit of the crystalline branch of helium, it can

be shown that lower bounds will be identical for the fluid branch with a smaller CP density. This can be seen from (6) if we change ρ_0 to $\alpha\,\rho_{\rm RCP}$ with $\alpha \geq 1$, so that the new residue would then be $\alpha^{1/3}\,A$ which is larger than the original residue A.

From the original (boson) London equation (6) a residue of $A = \pi^2/2^{1/3} \simeq 7.83$ was found. This residue was also assumed [18] in the generalized (fermion) London equation (9) to (11). Although the residue obeys the bounds (5), it generates energies deviating appreciably [20] from boson GFMC data [9]. In the Modified [20] London equation (6) the residue increases by a factor of two. With this modification the energy suddenly agrees rather well with the GFMC fluid data and the new residue coincides with the value extracted from experiment. However, we shall use this equation to represent not the fluid but rather the *crystalline branch* since it close-packs at the proper expected *regular* CP density.

To model the *fluid branch* energy we start with the exact low-density series and extrapolate them to some RCP density, smaller than the fcc or hexagonal CP density $\rho_0 = \sqrt{2}/c^3$. In addition to being well-behaved at intermediate densities, we observe that these extrapolations also obey the bounds (5). For the boson hard-sphere fluid the extrapolant form XI(x) (17) has the smaller mean-square-deviation from GFMC, a CP density $\rho = 0.78\,\rho_0$ which is thus nearer to the classical RCP value of 0.86 ρ_0, and a residue of 17.05 which is just slightly larger than Cole's experimental residue of 15.7 ± 0.6.

In Table 4 we summarize our results. For a fermion system with $\nu = 2$, the ground state energy (23) develops a RCP pole near the boson-hard-sphere RCP and around the expected classical RCP value. However, its residue of $A = 4.92$ is notably smaller than the experimental residue for a regular CP arrangement. The reason for this is probably that the fifth term in the series (21) may be a logarithmic instead of a power term. For fermions with $\nu = 4$ the extrapolant XII(x), Eq. (28), to $e_0^{-1/2}(x)$ produces energies with acceptable behavior in the range of densities from zero to the saturation density of nuclear matter. Unfortunately, it produces a packing density

of only $0.385\,\rho_0$. This may suggest that the number of terms considered in the series (24) and (25) are insufficient.

5. Conclusions.

To construct hard-sphere fluid equations-of-state all available coefficients in the low-density series expansion for the ground-state energy-per-particle of a many-fermion and many-boson system of particles were employed along with physical arguments related to close-packing densities and residues. These equations should be reasonably accurate from zero through physical densities as zero-order approximations in a quantum thermodynamic perturbation theory of real quantum fluids such as liquid-^{3}He, liquid-^{4}He, spin-polarized hydrogen (H$\downarrow$), nuclear and neutron matter, etc.

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Figure Captions

Figure 1. Dimensionless quantity $\varepsilon_0^{-1/2} \equiv \sqrt{2\pi\hbar^2\rho\, c\, N/m\, E} = [1 - (\rho/\rho_0)^{1/3}] \times \sqrt{1 + b\,(\rho/\rho_0)^{1/3}}$ for London equations (dashed curves) and for Modified London equations (full curves), as functions of ρ/ρ_0, for bosons (ML_∞), for fermions with $\nu = 2$ (ML_2) and for fermions with $\nu = 4$ (ML_4). GFMC data are for boson hard-spheres; open circles are data for fluid branch and full circles for solid branch.

Figure 2. Dimensionless quantity $\varepsilon_0^{-1/2} \equiv \sqrt{2\pi\hbar^2\rho\, c\, N/m\, E}$ for bosons ($\nu = \infty$), fermions with $\nu = 2$ and $\nu = 4$, as function of $(\rho/\rho_0)^{1/3}$, where $\rho_0 \equiv \sqrt{2}/c^3$. Extrapolations for the fluid branch are denoted by XI (full line) for bosons, by [3//2] (dashed line) for fermions with $\nu = 2$, and by XII (dot-dashed line) for fermions with $\nu = 4$, all as discussed in the text. Open circles are GFMC data for the fluid branch of boson hard-spheres. "Ladder" datapoints come from Ref. [26], while the "L-expansion" ones come from both Ref. [26] and [27]. The three arrows on the abscissa axis mark the saturation densities of nuclear matter ($\rho_s/\rho_0 \simeq 0.008$ with $c = 0.4$ fm), of liquid ^{3}He ($\rho_s/\rho_0 \simeq 0.108$ with $c = 2.1117$ Å) and of liquid ^{4}He ($\rho_s/\rho_0 \simeq 0.153$ with $c = 2.1463$ Å), respectivelly.

Table Captions

Table 1. C_i and F_i coefficients in (14) and (15), respectively.

Table 2. D_i and F_i coefficients in (20) and (21), respectively.

Table 3. D_i and F_i coefficients in (24) and (25), respectively.

Table 4. Energies per particle, E/N, in units of $(\hbar^2/mc^2)$, compared at the saturation density ρ_s/ρ_0, as well as at a particular close packing density $(\rho/\rho_0)_{\mathrm{CP}}$, and the residue A resulting from the method used. Boson hard-sphere energies are given at ^{4}He saturation density, fermion ($\nu = 2$) hard-sphere energies are given at ^{3}He saturation density and fermion ($\nu = 4$) hard-sphere energies are given at nuclear matter saturation density. The value of c for ^{3}He and ^{4}He is taken as the respective empirical S-wave scattering length, and for nuclear matter as the value 0.4 fm.

$\nu = \infty$	$i = 1$	2	3
C_i	4.81441778	19.65391518	"73.296"
F_i	-2.40720889	-9.826957589	"-27.956"

Table 1

$\nu = 2$	$i = 1$	2	3	4	5
D_i	0.353678	0.185537	0.384145	-0.024700	"-0.2655435"
F_i	-0.176833	-0.045863	-0.156677	0.109672	"0.130830"

Table 2

$\nu = 4$	$i = 1$	2	3	4
D_i	1.061033	0.556610	1.300620	-1.408598
F_i	-0.530517	0.143867	-0.5806558	-0.704299

Table 3

	case	E/N$[\hbar^2/mc^2]$	$(\rho/\rho_0)_{\rm CP}$	residue A
BOSONS	London	4.39	1	$\pi^2/2^{1/3} \simeq 7.83$
$\nu = \infty$	Mod. Lond.	6.62	1	$2\,\pi^2/2^{1/3} \simeq 15.67$
$\rho_s/\rho_0 = 0.153$	XI	6.65	0.776	17.05
$c = 2.1463$ Å	VMC	6.63	—	—
	GFMC	6.47	—	—
FERMIONS	London	2.683	1	$\pi^2/2^{1/3} \simeq 7.83$
$\nu = 2$	Mod. Lond.	3.224	1	$2\,\pi^2/2^{1/3} \simeq 15.67$
$\rho_s/\rho_0 = 0.1080$	[3//2]	3.180	0.732	4.97
$c = 2.1117$ Å	VFHNC	4.055	—	—
	Ex. Ladder	4.229	—	—
FERMIONS	London	0.1632	1	7.83
$\nu = 4$	Mod. Lond.	0.1738	1	15.67
$\rho_s/\rho_0 = 0.008$	XII	0.1651	0.385	12.83
$c = 0.4$ fm	VFHNC	0.1729	—	—

Table 4